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The effects of spatial coherence on intensity fluctuation distributions of Gaussian light

Abstract. The effects of finite sizes of source and detector on observed intensity fluctuations of Gaussian light are investigated both theoretically and experimentally. The theory allows a measurement of source size to be made using a single receiving aperture, rather than two as used in the Hanbury-Brown Twiss intensity interferometer.

In much recent work in statistical optics, temporal coherence properties have been investigated. Theoretical and experimental results have been obtained in which the effects of spatial coherence have been considered to be negligibly small due to the use of essentially point sources and detectors. A recent bibliography is given by Jakeman and Pike (1969). In real experiments, of course, the order of size of a 'point' source or detector before spatial coherence effects become measurable is of interest. Conversely, effects due to loss of spatial correlation can be used, as in the Michelson stellar interferometer, or the Hanbury-Brown Twiss intensity interferometer, to provide information about the source.

In recently reported quantitative work from this laboratory using intensity fluctuation spectroscopy of laser scattering to determine diffusion constants of protein molecules (Foord *et al.* 1970) it was necessary to calculate the effects of finite aperture sizes on the measurements in order to interpret fully the results obtained. This calculation together with supporting experimental data are presented in this paper. The results are applicable to the general problem of spatial integration over a detector surface of light from a quasimonochromatic Gaussian source. We prove this first below by showing that the mutual coherence of the scattered laser field between two points on the detector surface is identical to that arising from such a source. We then calculate the second moment of the intensity fluctuation distribution measured by a single detector, whose area is not small compared with a 'coherence area'. The result, which involves simply a double integral of the square of the Van Cittert–Zernicke mutual coherence function over the detector surface, can be thought of as providing the theory of a single-aperture intensity interferometer.

Consider the electric field at the point $\boldsymbol{r}$ due to the scattering of laser light of wave vector $\boldsymbol{k}_0$ from particles situated at the point $\boldsymbol{r}_j$ and moving with velocity $\boldsymbol{v}_j$. Let a fraction $n_\omega \delta\omega$ of the particles give rise to frequency shifts at $\boldsymbol{r}$ between ω and $\omega + \delta\omega$. The formula

$$\omega = \omega_0 - \boldsymbol{K}_j(\omega) \cdot \boldsymbol{v}_j \tag{1}$$

defines the frequency shift in terms of the scattering vector $\boldsymbol{K}_j(\omega)$ for these particles. The positive-frequency part of the scattered field may be written, by a simple calculation of phases, as

$$\mathscr{E}^+(\boldsymbol{r}, t) = \sum_{\omega} \alpha_{\omega}{}^+(\boldsymbol{r}) \exp(-\mathrm{i}\omega t) \tag{2}$$

where

$$\alpha_{\omega}{}^+(\boldsymbol{r}) = \mathscr{E}_0{}^+ \exp(\mathrm{i}\boldsymbol{k}_0 \cdot \boldsymbol{r}) \sum_j \exp\{\mathrm{i}\boldsymbol{K}_j(\omega) \cdot (\boldsymbol{r}-\boldsymbol{r}_j)\} \tag{3}$$

and represents the amplitude at the point $\boldsymbol{r}$ of the Fourier component of the field with frequency ω. The same constant $\mathscr{E}_0{}^+$ is associated with each particle and we consider one polarized component. The ensemble or long-time average

$$\langle \alpha_{\omega}{}^+(\boldsymbol{r})\alpha_{\omega'}{}^-(\boldsymbol{r}) \rangle \propto \sum_{ij} \langle \exp\{-\mathrm{i}\boldsymbol{K}_i(\omega) \cdot (\boldsymbol{r}-\boldsymbol{r}_i) + \mathrm{i}\boldsymbol{K}_j(\omega') \cdot (\boldsymbol{r}-\boldsymbol{r}_j)\} \rangle \tag{4}$$

vanishes unless $\omega = \omega'$, since otherwise i and j refer to particles in different sets and the phase differences are then large and random. When this equality is satisfied the exponent may be approximated in the far-field limit for a cylindrically symmetric system $\boldsymbol{r} \equiv (r, \psi, 0)$, $\boldsymbol{r}_i = (s_i, \phi_i\, z_i)$, where $\boldsymbol{k}_0$ is at an angle χ to the axis, by

$$-\frac{\mathrm{i}k_0}{2}\left\{\frac{r^2}{z_j} - \frac{r'^2}{z_i} + \frac{s_j{}^2}{z_j} - \frac{s_i{}^2}{z_i} - 2\frac{rs_j}{z_j}\cos(\phi_j - \psi) + 2\frac{r's_i}{z_i}\cos(\phi_i - \psi')\right\}$$
$$+(z_i - z_j)\sin\chi + \mathrm{i}k_0(s_j \sin\phi_j - s_i \sin\phi_i - r\sin\psi + r'\sin\psi')\cos\chi. \tag{5}$$

The random phase implicit in the last two terms of this expression means that contributions to (4) come only from the terms $i = j$. Using the density n_{ω} of particles giving rise to the frequency shift ω to convert the diagonal sum to an integral over the scattering volume leads to

$$\langle \alpha_{\omega}{}^+(\boldsymbol{r})\alpha_{\omega'}{}^-(\boldsymbol{r}') \rangle \propto n_{\omega}\delta_{\omega\omega'} \int_v s\,\mathrm{d}s\,\mathrm{d}\phi\,\mathrm{d}z \exp\left[\frac{\mathrm{i}k_0}{z}\left[s\{r\cos(\phi-\psi) - r'\cos(\phi-\psi')\}\right.\right.$$
$$\left.\left. - \tfrac{1}{2}(r^2 - r'^2)\right]\right] \exp\{\mathrm{i}k_0(r'\sin\psi' - r\sin\psi)\} \tag{6}$$

so that the normalized first-order autocorrelation function of the field,

$$\langle \mathscr{E}^+(\boldsymbol{r}, t)\mathscr{E}^-(\boldsymbol{r}, t') \rangle / \langle |\mathscr{E}^+(\boldsymbol{r}, t)|^2 \rangle$$

(mutual coherence function), is given from (2) and the Bessel integral (6) by

$$g^{(1)}(r, t; r', t') = \exp\left\{-\frac{\mathrm{i}k_0}{2z}(r^2 - r'^2)\right\} \frac{2J_1[\kappa\{r^2 + r'^2 - 2rr'\cos(\psi-\psi')\}^{1/2}]}{\kappa\{r^2 + r'^2 - 2rr'\cos(\psi-\psi')\}^{1/2}}$$
$$\times \sum_{\omega} n_{\omega} \exp\{-\mathrm{i}\omega(t-t')\} \tag{7}$$

where $\kappa = k_0S/2Z$, Z being the mean distance of the scattering volume from the plane $z = 0$ and S being the radius of the scattering volume. The form (7) for the field autocorrelation function reduces to the form given by Mandel and Wolf (1965) for a circular quasi-monochromatic Gaussian source when $n_{\omega} = \delta_{\omega\omega_0}$ and exhibits the property of "cross-spectral purity":

$$g^{(1)}(\boldsymbol{r}, t; \boldsymbol{r}', t') = g^{(1)}(\boldsymbol{r}, t; \boldsymbol{r}'t)g^{(1)}(\boldsymbol{r}, t; \boldsymbol{r}, t'). \tag{8}$$

The fluctuations seen by the detector are those of the square of the envelope of the field I integrated for a finite sample time T over the detector aperture A. The normalized temporal autocorrelation function of this quantity takes the form

$$g^{(2)}(\tau; T, A) = \frac{1}{A^2T^2}\left\langle \int_A \int_T \mathrm{d}\boldsymbol{r}\,\mathrm{d}t\, I(\boldsymbol{r}, t) \int_A \int_T \mathrm{d}\boldsymbol{r}\,\mathrm{d}t\, I(\boldsymbol{r}, t+\tau)\right\rangle \tag{9}$$

which for Gaussian light with the property (8) reduces to (Glauber 1963)

$$g^{(2)}(\tau; T, A) = 1 + \left\{\frac{1}{T^2}\int_{-T/2}^{T/2} \mathrm{d}t \int_{\tau-T/2}^{\tau+T/2} \mathrm{d}t' |g^{(1)}(r, t; r, t')|^2\right\} \times \left\{\frac{1}{A^2}\int_A \mathrm{d}\boldsymbol{r} \int_A \mathrm{d}\boldsymbol{r}' |g^{(1)}(r, t; r', t)|^2\right\}. \tag{10}$$

The effects of spatial coherence are given by the second of the factors appearing on the right-hand side of (9). Using the expansion for the Bessel function, this double integral over the detector aperture, assumed circular with radius R, may be expressed in the form

$$f(A) = \frac{4}{2\pi R^4}\int_0^R \int_0^R r_1 r_2\,\mathrm{d}r_1\,\mathrm{d}r_2 \int_0^{2\pi} \mathrm{d}\phi \times \sum_{s=0}^{\infty} \frac{(2+2s)!(-1)^s}{s!\{(s+1)!\}^2(s+2)!}(\tfrac{1}{2}\kappa)^{2s}(r_1{}^2 + r_2{}^2 - 2r_1 r_2 \cos\phi)^s. \tag{11}$$

Using the relations

$$\int_0^{2\pi} (1 - 2a\cos\phi + a^2)^n\,\mathrm{d}\phi = 2\pi \sum_{k=0}^{n} \binom{n}{k}^2 a^{2k} \tag{12}$$

and

$$\sum_{k=0}^{s} \binom{s}{k}^2 \frac{1}{(k+1)(s-k+1)} = \frac{(2s+2)!}{(s+1)(s+2)\{(s+1)!\}^2} \tag{13}$$

the integrals in (10) may be performed and the resulting expression reduced to the form

$$f(A) = \sum_{s=0}^{\infty} \left[\frac{(2s+2)!}{\{(s+1)!\}^2(s+2)!}\right]^2 (-1)^s(\tfrac{1}{2}\kappa R)^{2s}. \tag{14}$$

This function is plotted against κR in figure 1. It shows how the value of $g^{(2)}(\tau; T, A)$ decreases with the increasing fraction of a coherence area occupied by the detector surface, due to the averaging out of less and less correlated fluctuations. At $\tau = 0$ (zero time delay) this is the reduction in the second moment of the intensity fluctuation distribution. The theory has been applied to experimental measurements of the second factorial moment, $g^{(2)}(0; T, A)$, of light scattered from the protein haemocyanin undergoing Brownian motion. This takes the form, using known results (Jakeman and Pike 1968) for the T dependence of $g^{(2)}$ in equation (9),

$$g^{(2)}(0; T, A) - 1 = f(A)\left(\frac{1}{\gamma} - \frac{1}{2\gamma^2} + \frac{\mathrm{e}^{-2\gamma}}{2\gamma^2}\right) \tag{15}$$

where $\gamma = \Gamma T$ and Γ is the halfwidth at half height of the Lorentzian spectrum.

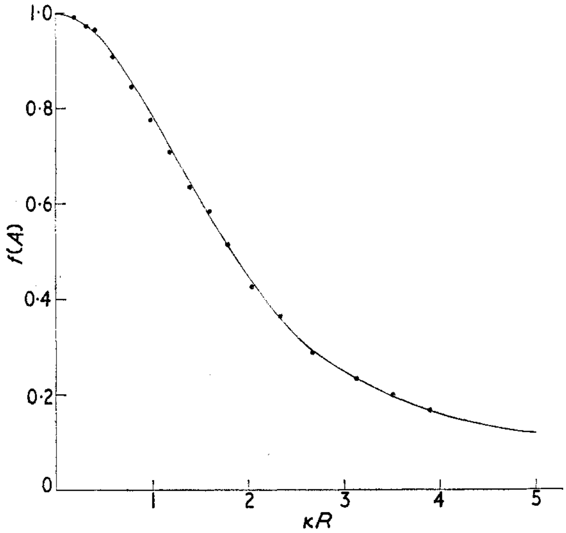

Figure 1. Theoretical and experimental decrease in the excess normalized second factorial moment of Gaussian–Lorentzian light due to spatial coherence effects. Curve: theory. Dots: experiment.

In making the measurement the light scattered through 90° from the protein undergoing Brownian motion was observed via a 0·05 mm radius circular aperture at the source. A second aperture on the detector was set a known distance away to determine κR. The unfocused laser beam (Spectra Physics 125) had a Gaussian profile with a radius to the $1/e^2$ power points of 0·64 mm. This was considerably larger than the first aperture and obviates 'limb darkening' effects. $f(A)$ was calculated from measurements of $g^{(2)}(0; T, A)$ using equation (15) over a range of values of κR from 0·2 to 3·9 giving excellent agreement with the theory (figure 1). The accuracy was such that as a single aperture intensity interferometer the experiment could be used to measure the diameter of apertures of the order to tens of microns with considerable precision.

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